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Statistics of addition spectra of independent quantum systems

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Abstract. Motivated by recent experiments on large quantum dots, we consider the energy spectrum in a system consisting of N particles distributed among $K < N$ independent subsystems, such that the energy of each subsystem is a quadratic function of the number of particles residing on it. On a large scale, the ground-state energy $E(N)$ of such a system grows quadratically with N, but in general there is no simple relation such as $E(N) = aN + bN^2$. The deviation of $E(N)$ from exact quadratic behaviour implies that its second difference (the inverse compressibility) $\chi_N \equiv E(N+1) - 2E(N) + E(N-1)$ is a fluctuating quantity. Regarding the numbers χ_N as values assumed by a certain random variable χ, we obtain a closed-form expression for its distribution $F(\chi)$. Its main feature is that the corresponding density $P(\chi) = \frac{\mathrm{d}F(\chi)}{\mathrm{d}\chi}$ has a maximum at the point $\chi = 0$. As $K \to \infty$ the density is Poissonian, namely, $P(\chi) \to \mathrm{e}^{-\chi}$.

1. Motivation

Statistics of spectra is an efficient tool for elucidating properties of various physical systems. So far, most of the effort has been focused on the study of energy levels of a system with a fixed number of particles. In this context, one of the central earlier results is that the spectral statistics of many-body systems such as complex nuclei agree with the predictions of random matrix theory [1, 2]. On the other extreme, it was found that level statistics of a single particle in a chaotic or disordered system also obeys a Wigner–Dyson statistics [3, 4].

Recently, experiments have been designed to obtain information on the statistics of the *addition spectra* of electrons in quantum dots [5]. The pertinent energy levels $E(N)$ are the ground-state energies of a system consisting of N electrons residing on a quantum dot, which is coupled capacitively to its environment.

Let us single out two properties of the addition spectra of quantum dots. The first one is that, on a large scale, the energy $E(N)$ grows quadratically with N, while the second one is a consequence of charge quantization, namely, there is, in general, no simple relation such as $E(N) = aN + bN^2$. In this context, an appropriate quantity whose statistics is of interest is then the inverse compressibility,

$$\chi_N \equiv E(N+1) - 2E(N) + E(N-1). \tag{1}$$

It is the deviation of $E(N)$ from exact quadratic behaviour which makes its second difference χ_N non-constant. Indeed, in a recent experiment on large quantum dots [6] it was found that the inverse compressibility vanishes for numerous values of electron number N.

In this work we study the statistics of the addition spectrum of a simple physical system with the two basic properties mentioned above. One example of such a system is motivated by considering the electrostatic energy of large quantum dots (although it should be mentioned that the model is too simple to describe the actual physics). To be specific, we have in mind a system of K metallic grains such that the number of electrons on the ith grain is n_i ($i = 0, 1, 2, \ldots, K-1$) and their total number is N. The electrostatic energy of the pertinent system is a bilinear form in the numbers n_i with a $K \times K$ matrix $w \equiv \frac{1}{2}C^{-1}$. Here C is a positive-definite symmetric matrix of capacitance and inductance coefficients. If the metallic grains are very far apart, the matrix C is nearly diagonal. Thus, we concentrate on the special case $C = \mathrm{diag}[C_i]$, for which the energy of the system is given by

$$E(N) = \min \sum_{i=0}^{K-1} \frac{1}{2C_i} n_i^2 \qquad \left(\text{subject to } \sum_{i=0}^{K-1} n_i = N\right). \tag{2}$$

The minimum in (2) is taken over all possible partitions n_i of N.

Another example is the energy of a system composed of K different harmonic oscillators, among which one distributes N spinless fermions. If there are n_i fermions on oscillator i (whose frequency is ω_i), then the energy of this oscillator (up to a constant) is $E_i = \hbar\omega_i n_i(n_i + 1)$, and hence the ground-state energy of the system is

$$E(N) = \min \sum_{i=0}^{K-1} E_i \qquad \left(\text{subject to } \sum_{i=0}^{K-1} n_i = N\right). \tag{3}$$

We will concentrate on the first example, which is borrowed from the electrostatics of quantum dots (2), and refer to the constants C_i as capacitors. Some remarks pertaining to the second example (the system of oscillators (3)) are also presented.

Regarding the numbers χ_N of (1) as values assumed by a certain random variable, the distribution of this random variable is the main focus of this work, which culminates in theorem 1, where we find a closed-form expression for the distribution.

The problem of elucidating the (addition) spectral statistics of a *a many-body* system, consisting of several independent subsystems (whose dependence of E on n_i is known), looks deceptively simple. As will be evident shortly, this is not the case, and finding the distribution in question is quite a non-trivial task. Note that, even for a *single-particle system* composed of several independent subsystems (e.g. a system of a particle in several boxes), the derivation of level statistics requires a large degree of mathematical effort [7]. The rest of the paper is therefore devoted to a rigorous derivation of our main results.

2. Formalism

Definition 1. Let $(\theta_n)_{n=1}^{\infty}$ be a sequence of real numbers and F a distribution function. The sequence (θ_n) is *asymptotically F-distributed* if

$$\frac{|\{1 \leqslant n \leqslant M : \theta_n \leqslant x\}|}{M} \underset{M\to\infty}{\longrightarrow} F(x)$$

for every continuity point x of F (where $|S|$ denotes the cardinality of a finite set S).

An equivalent condition is the following. Denote by δ_t the point mass at t, and let μ be the probability measure corresponding to the distribution F (namely, $\mu(A) = \int 1_A \,\mathrm{d}F(x)$ for any Borel set A). Then (θ_n) is asymptotically F-distributed if

$$\frac{1}{M}(\delta_{\theta_1} + \delta_{\theta_2} + \cdots + \delta_{\theta_M}) \underset{M\to\infty}{\longrightarrow} \mu$$

(the convergence being in the weak*-topology).

The notion of asymptotic distribution has a stronger version whereby, instead of requiring only that initial pieces of the sequence behave in a certain way, we require this to happen for any large finite portion of the sequence. This leads to the following.

Definition 2. In the set-up of definition 1, (θ_n) is *asymptotically well F-distributed* if

$$\frac{|\{L < n \leqslant M : \theta_n \leqslant x\}|}{M-L} \underset{M-L\to\infty}{\longrightarrow} F(x)$$

for every continuity point x of F.

Recall that the *density* of a set $A \subseteq \mathbb{N}$ is given by

$$\mathrm{D}(A) = \lim_{M\to\infty} \frac{|A \cap [1, M]|}{M}$$

if the limits exists. If, moreover, the limit

$$\mathrm{BD}(A) = \lim_{M-L\to\infty} \frac{|A \cap (L, M]|}{M-L}$$

exists (in which case it is certainly the same as $\mathrm{D}(A)$), then it is called the *Banach density* of A (cf [9, p 72]).

The following lemma is routine.

Lemma 1. Let $(\theta_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose $\mathbb{N} = \bigcup_{j=1}^{r} A_j$, where the union is disjoint. Let $(\theta_n^{(j)})_{n=1}^{\infty}$ be the subsequence of (θ_n), consisting of those elements θ_n with $n \in A_j$, $1 \leqslant j \leqslant r$.

(1) If each $(\theta_n^{(j)})$ is asymptotically F_j-distributed for some distribution functions F_j, $1 \leqslant j \leqslant r$, and $\mathrm{D}(A_j) = d_j$, $1 \leqslant j \leqslant r$, then (θ_n) is asymptotically F-distributed, where $F = \sum_{j=1}^{r} d_j F_j$.

(2) If each $(\theta_n^{(j)})$ is asymptotically well F_j-distributed and $\mathrm{BD}(A_j) = d_j$, then (θ_n) is asymptotically well F-distributed.

Obviously, a general sequence on the line does not have to be asymptotically distributed according to some distribution function, but one would expect it of sufficiently 'regular' bounded sequences. In our case, one might expect χ_N to be distributed according to some distribution function corresponding to a measure centred at about $1/C$. However, this is not the case. In fact, the measure in question is supported on a finite interval, and is a convex combination of an absolutely continuous measure with decreasing density function on some interval $[0, a]$ and the point mass δ_a at the right end a of that interval.

We have defined $E(N)$ indirectly by means of the following.

Problem 1. For each non-negative integer N, find non-negative integers $n_0, n_1, \ldots, n_{K-1}$, satisfying $n_0 + n_1 + \cdots + n_{K-1} = N$, for which $\sum_{i=0}^{K-1} \frac{1}{2C_i} \cdot n_i^2$ is minimal.

It turns out that this problem is intimately related to a second optimization problem. Put $w_i = \frac{1}{2C_i}$, $0 \leqslant i \leqslant K-1$, and let Δ denote the set of all positive odd multiples of the numbers $\frac{1}{2C_i}$:

$$\Delta = \{w_0, 3w_0, 5w_0, \ldots, w_1, 3w_1, 5w_1, \ldots, w_{K-1}, 3w_{K-1}, 5w_{K-1}, \ldots\}.$$

Here we treat Δ as a multiset, or a sequence, in the sense that if some elements appear in this representation of Δ more than once (which occurs iff some ratio w_i/w_j is a rational number with odd numerator and denominator), then we consider Δ as having several copies of these numbers.

Problem 2. For each non-negative integer N, minimize $\sum_{m=1}^{N} \delta_m$, where $\delta_1, \delta_2, \ldots, \delta_N$ range over all distinct N-tuples in Δ.

Note that, if an element appears several times in Δ, it is allowed to appear the same number of times in the sum as well.

Let us demonstrate the equivalence of the two problems. Given the sum $\sum_{i=0}^{K-1} w_i \cdot n_i^2$, we may use the equality $w_i \cdot n_i^2 = w_i + 3w_i + 5w_i + \cdots + (2n_i - 1)w_i$ to see that any feasible value for the objective function of the first problem is a feasible value for the objective function of the second problem as well. On the other hand, solving problem 2 is trivial. Namely, one minimizes the sum there simply by taking the N least elements of the set Δ. In particular, for each i, the multiples of w_i present in the optimal solution will be all odd multiples $w_i, 3w_i, 5w_i, \ldots$ up to some $(2n_i - 1)w_i$. Thus, the optimal solution of problem 2 also yields the optimal solution of problem 1. We note in passing that this discussion also shows that the minimum (for each of the problems) is obtained at a unique point unless Δ contains multiple elements. (However, we shall always refer to *the* optimal solution, even when there may be several.)

A simple consequence of the above is as follows.

Proposition 1. Let $\boldsymbol{n} = (n_i)_{i=0}^{K-1}$ be the optimal solution of problem 1 for some value of N. Then the optimal solution of problem 1, with $N + 1$ instead of N, is $\boldsymbol{n}' = (n_i')_{i=0}^{K-1}$, where $n_j' = n_j + 1$ for some $0 \leqslant j \leqslant K - 1$ and $n_i' = n_i$ for $i \neq j$.

Remark. It is convenient to comment here on the effect of a certain change in the original problem would make. One may consider the energies E_i to be $w_i n_i (n_i + 1)$ instead of $w_i n_i^2$. This would change Δ to be the set of all even multiples of the w_i's. Obviously, this would leave intact the equivalence of problems 1 and 2. One can check that this would have also no effect on theorems 1 and 2 below.

To formulate our main result we need a few definitions and notations. Real numbers $\theta_1, \theta_2, \ldots, \theta_r$ are independent over $\mathbb{Q}$ if, considered as vectors in the vector space $\mathbb{R}$ over the field $\mathbb{Q}$, they are linearly independent. Equivalently, this is the case if the equality $m_1\theta_1 + m_2\theta_2 + \cdots + m_r\theta_r = 0$ for integer $m_1, m_2, \ldots, m_r$ implies $m_1 = m_2 = \cdots = m_r = 0$. Considering the actual physical system (a collection of metallic grains), it is reasonable to assume that the capacitors C_i are random, so that generically they are independent over $\mathbb{Q}$. Without loss of generality we may rearrange the K capacitors such that $C_0 = \max_{0 \leqslant i \leqslant K-1} C_i$. It is also useful to divide all the capacitors by the largest one, so that the scaled capacitors $c_i \equiv C_i / C_0$ with $1 = c_0 > c_1, c_2 \ldots, c_{K-1}$ are dimensionless. Finally, set $s = c_0 + c_1 + \cdots + c_{K-1}$.

Now we formulate our main results.

Theorem 1. Suppose $C_0, C_1, \ldots, C_{K-1}$ are independent over $\mathbb{Q}$. Then the sequence $(\chi_N)_{N=1}^{\infty}$ is asymptotically F-distributed, where the distribution F is given by either of the following two representations:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - \dfrac{1}{s} \displaystyle\sum_{i=0}^{K-1} c_i \prod_{\substack{j=0 \\ j \neq i}}^{K-1} \left(1 - \dfrac{x}{2w_j}\right) & 0 \leqslant x < 2w_0 \\ 1 & 2w_0 \leqslant x \end{cases} \tag{4}$$

$$= \begin{cases} 0 & x < 0 \\ 1 - \dfrac{1}{s} \displaystyle\sum_{S \subseteq \{1,\ldots,K-1\}} (|S|+1) \prod_{i \in S} c_i \prod_{i \notin S} (1-c_i) \cdot \left(1 - \dfrac{x}{2w_0}\right)^{|S|} & 0 \leqslant x < 2w_0 \\ 1 & 2w_0 \leqslant x. \end{cases} \tag{5}$$

It is not immediately obvious from the formulae, but F has one discontinuity, namely at the point $2w_0$. The reason is that, as the elements of Δ are all odd multiples of the w_i's, and as w_0 is the smallest of the w_i's, it happens occasionally that there is no odd multiple of $w_1, \ldots, w_{K-1}$ between two consecutive multiples of w_0. The size of the atom at $2w_0$ is $\frac{1}{s} \cdot \prod_{i=1}^{K-1}(1-c_i)$. This is easily explained intuitively. In fact, the 'density' of odd multiples of w_i is c_i times the same density for multiples of w_0. Hence, the 'probability' that an interval of the form $[(2n-1)w_0, (2n+1)w_0)$ does not contain an odd multiple of w_i is $1-c_i$. Assuming that the 'events' of containing different w_i's are independent, we conclude that the proportion of multiples of w_0 in Δ whose successors are also such is $\prod_{i=1}^{K-1}(1-c_i)$. Since the proportion of multiples of w_0 in Δ is $\frac{1}{s}$, we arrive at the required expression for the size of the atom.

Now we would like to study the asymptotic of the distances between consecutive elements of Δ as the number of capacitors grows. Obviously, as this happens, the distances become smaller. More precisely, on average we have $\frac{1}{2w_j}$ odd multiples of each w_j in each unit interval, and hence we have there $\sum_{j=0}^{K-1} \frac{1}{2w_j} = \frac{s}{2w_0}$ elements of Δ altogether. Hence, the average distance between consecutive elements is $\frac{2w_0}{s}$. To understand the asymptotics of the gaps, it makes sense therefore to normalize them so as to have mean 1. Thus, we multiply the distances by $\frac{s}{2w_0}$, and ask about the asymptotic behaviour.

Theorem 2. Suppose the capacitances $C_0, C_1, \ldots$ are chosen uniformly and independently in $[0, 1]$. For each K, let F_K denote the distribution corresponding to the normalized gaps when taking into account the first K capacitors only. Then, with probability 1, the distributions F_K converge to an Exp(1) distribution function.

Remark. As will be seen in the proof, we actually use much less to prove theorem 2 than is required by the conditions of the theorem. Namely, we need the capacitances C_i to be linearly independent over $\mathbb{Q}$, and that they do not form a fast diminishing sequence.

It is worthwhile mentioning that this type of 'Poissonian' asymptotic behaviour of consecutive gaps is typical. For example, this is the case for uniformly selected numbers in [0, 1], and is conjectured to be the case in other interesting cases as well (see, for example, [11, 12] and references therein).

In the course of the proof, we shall use the notion of uniform distribution modulo 1 and a few basic results relating to it. (The reader is referred to Kuipers and Niederreiter [10] for more information.) A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is *uniformly distributed modulo* 1 if

$$\frac{|\{1 \leqslant n \leqslant M : a \leqslant \{x_n\} < b\}|}{M} \xrightarrow[M\to\infty]{} b - a \qquad 0 \leqslant a < b \leqslant 1$$

where $\{t\}$ is the fractional part of a real number t. In terms of definition 1, (x_n) is uniformly distributed modulo 1 if and only if the sequence $(\{x_n\})$ of fractional parts is F-distributed, where F is the distribution function of the uniform distribution on $[0, 1]$:

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leqslant x \leqslant 1 \\ 1 & x > 1. \end{cases}$$

The generalization of the notion of an asymptotically F-distributed sequence to that of an asymptotically well F-distributed sequence clearly carries over to our case. Instead of requiring only that the dispersion of large initial pieces of the sequence becomes more and more even, we require this to happen at arbitrary locations. This version is termed *well distribution*. Thus, $(x_n)_{n=1}^{\infty}$ is *well distributed modulo* 1 if

$$\frac{|\{L < n \leqslant M : a \leqslant \{x_n\} < b\}|}{M - L} \underset{M-L\to\infty}{\longrightarrow} b - a \qquad 0 \leqslant a < b \leqslant 1.$$

Both notions have a multidimensional analogue. A sequence $(\boldsymbol{x}_n)_{n=1}^{\infty}$ in $\mathbb{R}^s$ is *uniformly distributed modulo 1 in* $\mathbb{R}^s$ if

$$\frac{|\{1 \leqslant n \leqslant N : \boldsymbol{a} \leqslant \{\boldsymbol{x}_n\} < \boldsymbol{b}\}|}{N} \underset{N\to\infty}{\longrightarrow} \prod_{i=1}^{s}(b_i - a_i) \qquad \boldsymbol{0} \leqslant \boldsymbol{a} < \boldsymbol{b} \leqslant \boldsymbol{1}$$

where inequalities between vectors in $\mathbb{R}^s$ are to be understood component-wise, $\boldsymbol{0} = (0, 0, \ldots, 0) \in \mathbb{R}^s$, $\boldsymbol{a} = (a_1, a_2, \ldots, a_s)$, etc.

Perhaps the most basic example of a sequence which is uniformly distributed modulo 1 is $(n\alpha)_{n=1}^{\infty}$, where α is an arbitrary irrational. In the multidimensional case, the sequence $(n\alpha_1, n\alpha_2, \ldots, n\alpha_s)$ is uniformly distributed modulo 1 in $\mathbb{R}^s$ if and only if the numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_s$ are linearly independent over $\mathbb{Q}$. Moreover, in this case uniform distribution implies well distribution (cf [10, example 1.6.1, exercise 1.6.14]).

Given a partition $\mathbb{N} = \bigcup_{j=1}^{l} A_j$ and positive integers r_j, $j = 1, \ldots, l$, we define the $(r_j)_{j=1}^{l}$*-inflation* of the given partition as the partition of $\mathbb{N}$ obtained by inflating each element of each of the sets A_j into r_j elements. More precisely, we construct sets B_j, $j = 1, \ldots, l$, as follows. For a positive integer i, let $f(i) = j$ if $i \in A_j$. Given any positive integer n, let m be defined by $\sum_{i=1}^{m-1} f(i) < m \leqslant \sum_{i=1}^{m} f(i)$. Let $n \in B_j$ if $m \in A_j$. The following lemma is routine.

Lemma 2. In this set-up:

(1) if $\mathrm{D}(A_j) = d_j$, $1 \leqslant j \leqslant l$, then $\mathrm{D}(B_j) = \frac{r_j d_j}{\sum_{i=1}^{l} r_i d_i}$;

(2) if $\mathrm{BD}(A_j) = d_j$, $1 \leqslant j \leqslant l$, then $\mathrm{BD}(B_j) = \frac{r_j d_j}{\sum_{i=1}^{l} r_i d_i}$.

Proof of theorem 1. Between any two consecutive odd multiples of w_0, there is at most one odd multiple of each w_j, $1 \leqslant j \leqslant K - 1$. In fact, one easily verifies that, given a positive integer m, there is an odd multiple of w_j between $(2m - 1)w_0$ and $(2m + 1)w_0$, namely there exists an integer n with

$$(2m - 1)w_0 \leqslant (2n - 1)w_j < (2m + 1)w_0 \tag{6}$$

if and only if

$$mc_j \in \left(\frac{1 - c_j}{2}, \frac{1 + c_j}{2}\right] (\mathrm{mod}\, 1). \tag{7}$$

Moreover, the relative position of $(2n-1)w_j$ within the interval $[(2m-1)w_0, (2m+1)w_0)$ is the same, but in the opposite direction, as that of $mc_j (\mathrm{mod}\, 1)$ within the interval $(\frac{1-c_j}{2}, \frac{1+c_j}{2}]$, that is

$$(2n - 1)w_j = \alpha \cdot (2m - 1)w_0 + (1 - \alpha) \cdot (2m + 1)w_0 \qquad (0 < \alpha \leqslant 1) \tag{8}$$

if and only if

$$mc_j \equiv (1-\alpha)\cdot\frac{1-c_j}{2} + \alpha\cdot\frac{1+c_j}{2} (\text{mod}\, 1). \tag{9}$$

Next we define a partition of $\mathbb{N}$ as follows. Write the elements of Δ in ascending order: $\Delta = \{\delta_1 < \delta_2 < \delta_3 < \cdots\}$. Given $n \in \mathbb{N}$, let $S \subseteq \{1, 2, \ldots, K-1\}$ denote the set of all those j's such that the unique interval of the form $[(2m-1)w_0, (2m+1)w_0)$ containing δ_n contains an odd multiple of w_j. The set of all integers n giving rise in this way to any set S is denoted by B_S. Consider the partition $\mathbb{N} = \bigcup_{S\subseteq\{1,2,\ldots,K-1\}} B_S$. To prove the theorem using lemma 1, we have to find the Banach densities of the sets B_S and the asymptotic distribution of the corresponding subsequences $(\chi_n)_{n\in B_S}$ of χ_n.

The partition of $\mathbb{N}$ into sets of the form B_S is obtained as an inflation of a somewhat more straightforward partition. In fact, let S be any subset of $\{1, 2, \ldots, K-1\}$. Denote by A_S the set of those positive integers n for which the interval $[(2n-1)w_0, (2n+1)w_0)$ contains odd multiples of w_j for $j \in S$ and does not contain such multiples of the other w_j's. Then $\mathbb{N} = \bigcup_{S\subseteq\{1,\ldots,K-1\}} A_S$ is a partition, and its $(|S|+1)_{S\subseteq\{1,2,\ldots,K-1\}}$-inflation yields the partition $\mathbb{N} = \bigcup_{S\subseteq\{1,2,\ldots,K-1\}} B_S$.

In view of the equivalence of (6) and (7), A_S is the set of those n's for which $nc_j \in (\frac{1-c_j}{2}, \frac{1+c_j}{2}]$ for $j \in S$ and $nc_j \notin (\frac{1-c_j}{2}, \frac{1+c_j}{2}]$ for $j \notin S$. By the conditions of the theorem, the numbers $1, c_1, \ldots, c_{K-1}$ are linearly independent over $\mathbb{Q}$, and hence the sequence $\boldsymbol{c} = (nc_1, nc_2, \ldots, nc_{K-1})_{n=1}^{\infty}$ is well distributed modulo 1 in $\mathbb{R}^{K-1}$. This means that

$$\mathrm{D}(A_S) = \mathrm{BD}(A_S) = \prod_{i\in S} c_i \prod_{i\notin S}(1-c_i). \tag{10}$$

Denote the right-hand side of (10) by p_S. In view of the above and lemma 2, this implies

$$\mathrm{D}(B_S) = \mathrm{BD}(B_S) = \frac{(|S|+1)p_S}{\sum_{T\subseteq\{1,2,\ldots,K-1\}}(|T|+1)p_T}. \tag{11}$$

The denominator on the right-hand side can be given a simpler form. In fact, let X_i, $i = 1, 2, \ldots, K-1$, be independent random variables with $X_i \sim B(1, c_i)$, and $X = \sum_{i=1}^{K-1} X_i$. Then:

$$\sum_{T\subseteq\{1,2,\ldots,K-1\}} (|T|+1)p_T = E(X+1) = 1 + c_1 + \cdots + c_{K-1} = s. \tag{12}$$

Hence:

$$\mathrm{BD}(B_S) = \frac{(|S|+1)p_S}{s}. \tag{13}$$

Let S be an arbitrary fixed subset of $\{1, 2, \ldots, K-1\}$, say $S = \{1, 2, \ldots, l\}$, where $0 \leqslant l \leqslant K-1$. If $n \in A_S$, then there exist odd integers $a_{1n}, a_{2n}, \ldots, a_{ln}$ such that $a_{jn}w_j \in [(2n-1)w_0, (2n+1)w_0)$. Put:

$$v_n = (a_{1n}w_1, a_{2n}w_2, \ldots, a_{ln}w_l) - (2n-1)w_0 \cdot (1, 1, \ldots, 1) \in [0, 2w_0)^l \qquad n \in A_S.$$

By the equivalence of (8) and (9), the sequence $(v_n)_{n\in A_S}$ is well distributed modulo $2w_0$ in $\mathbb{R}^l$. Now each v_n gives rise to $l+1$ terms of $(\chi_n)_{n\in B_S}$, as follows. Let $v_n^{(1)} \leqslant v_n^{(2)} \leqslant \cdots \leqslant v_n^{(l)}$ be all coordinates of v_n in ascending order. Set:

$$u_n = (v_n^{(1)}, v_n^{(2)} - v_n^{(1)}, \ldots, v_n^{(l)} - v_n^{(l-1)}, 2w_0 - v_n^{(l)}) \qquad n \in A_S.$$

The sequence $(\chi_n)_{n\in B_S}$ consists of all coordinates of all vectors u_n. Now we use the fact that if $X_1, X_2, \ldots, X_r$ are independent random variables, distributed $\mathrm{U}(0, h)$, and

$X^{(1)}, X^{(2)}, \ldots, X^{(r)}$ are the corresponding order statistics, then each of the random variables $X^{(1)}, X^{(2)} - X^{(1)}, \ldots, X^{(r)} - X^{(r-1)}, h - X^{(r)}$ has the distribution function defined by $G(x) = 1 - (x/h)^r$ for $0 \leqslant x \leqslant h$ (which follows as a special case from [8, p 42, exercise 23]). Consequently, for each $1 \leqslant j \leqslant l + 1$, the sequence given by the jth coordinate of all vectors u_n, $n \in A_S$, is asymptotically well G_1-distributed, where $G_1(x) = 1 - (x/2w_0)^l$ for $0 \leqslant x \leqslant 2w_0$. Hence, the sequence $(\chi_n)_{n\in B_S}$ is asymptotically well G_1-distributed. Combined with (13), it proves (5).

We shall indicate only briefly the proof of (4), which is quite simpler. This time, we split (χ_n) into a union of subsequences $(\chi_n^{(i)})$, $0 \leqslant i \leqslant K-1$, by putting χ_n in the sequence $\chi_n^{(i)}$ if δ_n is a multiple of w_i. Clearly, the proportion of terms of (χ_n) belonging to $(\chi_n^{(i)})$ is c_i/s. Next, consider the minimal odd multiples of all w_j's which are larger than δ_n. The minimum of these K numbers is δ_{n+1}. For each $j \neq i$, the distance from δ_n to the minimal odd multiple of w_j following δ_n is 'distributed' $\mathrm{U}(0, 2w_j)$. (For $i = 0$ it is also possible that the next term will be again a multiple of w_0.) The linear independence of the C_i's over $\mathbb{Q}$ implies that these $K - 1$ distances are (statistically) independent, so that their minimum is distributed according to the function $G_2(x) = 1 - \prod_{\substack{j=0\\ j\neq i}}^{K-1}(1 - \frac{x}{2w_j})$ on the interval $[0, 2w_0)$. These considerations can be formalized to prove (4). This completes the proof. □

Remark. It is possible to shorten the proof by proving directly the equality of the right-hand sides of (4) and (5). In fact, it is easy to integrate both forms with respect to x; the equality of the resulting expressions follows easily from the binomial theorem. We have chosen the long way, as it is more instructive.

Proof of theorem 2. The distribution F_K is obtained from that in theorem 1 by stretching by the constant factor $\frac{s}{2w_0}$. Hence:

$$F_K(x) = \begin{cases} 0 & x < 0 \\ 1 - \dfrac{1}{s}\displaystyle\sum_{i=0}^{K-1} c_i \prod_{\substack{j=0\\ j\neq i}}^{K-1}\left(1 - \frac{c_j x}{s}\right) & 0 \leqslant x < s \\ 1 & s \leqslant x. \end{cases} \tag{14}$$

Note that some of the values appearing on the right-hand side depend on K implicitly. Namely, since w_0 is assumed in theorem 1 to be the least w_i, each time a C_i is selected which is larger than all the heretofore selected C_j's, we have to rearrange the C_j's, thus changing w_0 and the c_j's. We have to show that

$$F_K(x) \underset{K\to\infty}{\longrightarrow} 1 - \mathrm{e}^{-x} \qquad x \geqslant 0. \tag{15}$$

Indeed, fix $x \geqslant 0$. Since

$$s = c_0 + c_1 + \cdots + c_{K-1} = \frac{C_0 + C_1 + \cdots + C_{K-1}}{C_0} \geqslant C_0 + C_1 + \cdots + C_{K-1} \tag{16}$$

and the C_i's are independent and uniformly distributed in $[0, 1]$, we have

$$s \xrightarrow[K\to\infty]{\text{a.s.}} \infty. \tag{17}$$

Hence, with probability 1, for sufficiently large K we have

$$F_K(x) = 1 - \frac{1}{s}\sum_{i=0}^{K-1} c_i \prod_{\substack{j=0\\ j\neq i}}^{K-1}\left(1 - \frac{x}{2w_j}\right). \tag{18}$$

Thus, to prove (15) we need to show that

$$\frac{1}{s}\sum_{i=0}^{K-1} c_i \prod_{\substack{j=0\\ j\neq i}}^{K-1}\left(1-\frac{c_j x}{s}\right) \xrightarrow[K\to\infty]{\text{a.s.}} \mathrm{e}^{-x} \qquad x \geqslant 0. \tag{19}$$

Now, on the one hand, using the inequality

$$1-t \leqslant \mathrm{e}^{-t} \qquad t \in \mathbb{R}$$

we have

$$\prod_{\substack{j=0\\ j\neq i}}^{K-1}\left(1-\frac{c_j x}{s}\right) \leqslant \mathrm{e}^{-x\sum_{\substack{j=0\\ j\neq i}}^{K-1}\frac{c_j}{s}} \leqslant \mathrm{e}^{-x+x/s} \qquad i=0,1,\ldots,K-1$$

and therefore

$$\frac{1}{s}\sum_{i=0}^{K-1} c_i \prod_{\substack{j=0\\ j\neq i}}^{K-1}\left(1-\frac{c_j x}{s}\right) \leqslant \frac{1}{s}\sum_{i=0}^{K-1} c_i \mathrm{e}^{-x+x/s} = \mathrm{e}^{-x+x/s} \xrightarrow[K\to\infty]{\text{a.s.}} \mathrm{e}^{-x}. \tag{20}$$

On the other hand, as $t \to 0$ we have

$$\mathrm{e}^{-(t+t^2)} = 1-(t+t^2)+\frac{(t+t^2)^2}{2}+\mathrm{O}(t^3) = 1-t-\frac{t^2}{2}+\mathrm{O}(t^3)$$

so that for all t in some sufficiently small neighbourhood of 0

$$\mathrm{e}^{-(t+t^2)} \leqslant 1-t.$$

Consequently:

$$\prod_{\substack{j=0\\ j\neq i}}^{K-1}\left(1-\frac{c_j x}{s}\right) \geqslant \mathrm{e}^{-x\sum_{\substack{j=0\\ j\neq i}}^{K-1}\frac{c_j}{s}-x^2\sum_{\substack{j=0\\ j\neq i}}^{K-1}\frac{c_j^2}{s^2}} \geqslant \mathrm{e}^{-x-Kx^2/s^2}. \tag{21}$$

Obviously, with probability 1, s grows linearly with K, namely for all sufficiently large K we have $s \geqslant aK$ for a suitably chosen $a > 0$. (In fact, any $a < \frac{1}{2}$ will do.) By (21):

$$\prod_{\substack{j=0\\ j\neq i}}^{K-1}\left(1-\frac{c_j x}{s}\right) \geqslant \mathrm{e}^{-x-Kx^2/s^2} \xrightarrow[K\to\infty]{\text{a.s.}} \mathrm{e}^{-x}. \tag{22}$$

From (20) and (22) it follows that

$$\frac{1}{s}\sum_{i=0}^{K-1} c_i \prod_{\substack{j=0\\ j\neq i}}^{K-1}\left(1-\frac{c_j x}{s}\right) \xrightarrow[K\to\infty]{\text{a.s.}} \mathrm{e}^{-x} \tag{23}$$

which completes the proof. □

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